

## Semichirals and Four Dimensional Geometry

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Semichiral  $(2,2)$  sigma models with  $4d$  target space are discussed. A novel description in  $(1,1)$  superspace allows an analysis of possible extended supersymmetries. It is argued that a manifest semichiral realization of an extra supersymmetry is only possible for hyperkähler target geometry. A semichiral formulation of the  $SU(2) \otimes U(1)$  WZW model is seemingly a counterexample to this. After deriving the extra supersymmetries of this model in  $(1,1)$  superspace it is shown that they cannot be lifted to transformations of semichirals in  $(2,2)$  superspace.

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## 1. Introduction

A problem discussed in a recent paper [1] is reviewed and expounded on here: Under quite general conditions, a realization of additional supersymmetries on the  $(2,2)$  fields of a semichiral sigma model with  $4d$  target space requires hyperkähler target space. Nevertheless the  $4d$  WZW model representing  $SU(2) \otimes U(1)$  has  $(4,4)$  supersymmetry in a (twisted) chiral formulation and this formulation has a semichiral dual with non-trivial torsion.

In this paper we argue that the resolution of this seeming contradiction is that the  $(4,4)$  is indeed present in the  $(1,1)$  superspace formulation of the semichiral model, but it is incompatible with the chirality conditions of the semichiral fields and hence cannot be lifted to a  $(2,2)$  semichiral formulation. This is shown using a novel  $(1,1)$  description where the auxiliary spinors are kept and the additional supersymmetries act on these as well as on the coordinate fields.

The plan of the paper is as follows: In section 2. the necessary background concerning superspace and complex geometry is given. Section 3. states the problem with a realization of extra supersymmetries on semichiral fields. In section 4. duality of the  $SU(2) \otimes U(1)$  WZW model is presented as well as an analysis of how the Legendre transform relates the condition for  $(4,4)$  in a  $4d$  (twisted) chiral sigma model with certain isometries to the hyperkähler condition of its semichiral dual. In particular we find a striking formulation of the latter in the natural dual coordinates. The subsection is not directly relevant for the main line of argument, but is included for the sake of completeness. Section 5. contains the derivation of the novel formulation keeping the auxiliary spinors, examples of transformations satisfying the “lifting” condition as well as the derivation of the extra supersymmetries for the semichiral  $SU(2) \otimes U(1)$  WZW model and the final negative result on their lifting.

## 2. Background

### 2.1 $(2,2)$ superspace

Generalized Kähler Geometry (GKG), [2], is the target space geometry of a two-dimensional non-linear sigma model with a generalized Kähler potential, [7],  $K$  that is a function of chiral, twisted chiral, left and right semichiral, [4],  $(2,2)$  superfields :

$$S = \int d^2\xi d^2\theta d^2\bar{\theta} K(\phi, \chi, \ell, r, \dots) \quad (2.1)$$

where the dots indicate the complex conjugate fields and  $(\xi^\mu, \theta^\pm, \bar{\theta}^\pm)$  coordinatize  $(2,2)$  superspace. The chirality conditions obeyed by the fields are:

$$\bar{\mathbb{D}}_\pm \phi = 0, \quad \bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = 0, \quad \bar{\mathbb{D}}_+ \ell = 0, \quad \bar{\mathbb{D}}_- r = 0, \quad (2.2)$$

and the complex conjugate relations. The  $(2,2)$  algebra is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = \iota \partial_\pm \quad (2.3)$$

To describe a sigma model, the number of left and right semichiral fields must be equal. We shall be interested in the minimal symplectic case; one left and one right field.

$$K = K(\ell, \bar{\ell}, r, \bar{r}) =: K(L, R). \quad (2.4)$$

The corresponding target space is four (real) dimensional.

## 2.2 (1,1) superspace

The reduction to (1,1) superspace reveals the geometry of the model and is achieved using

$$\mathbb{D}_\pm = \frac{1}{2}(D_\pm - \iota Q_\pm) , \quad \bar{\mathbb{D}}_\pm = \frac{1}{2}(D_\pm + \iota Q_\pm) , \quad (2.5)$$

where  $D_\pm$  are the (1,1) spinor derivative and  $Q_\pm$  the nonmanifest second supersymmetry generators. The action is reduced as

$$S \rightarrow \int d^2\xi D^2 Q^2 K| = \int d^2\xi d^2\theta (Q^2 K)| \quad (2.6)$$

where  $Q^2 := Q_+ Q_-$  et.c., a horizontal bar denotes setting the second  $\theta$  to zero, and the (1,1) components are defined as

$$\ell| = \ell , \quad Q_+ \ell| = \iota D_+ \ell , \quad Q_- \ell| =: \psi_- , \quad r| = r , \quad Q_+ r| =: \psi_+ , \quad Q_- r| = \iota D_- r \quad (2.7)$$

and their complex conjugate. The fields  $\psi_\pm$  are auxiliary (1,1) superfields and have to be integrated out to display the (1,1) sigma model:

$$S = \int d^2\xi d^2\theta D_+ X^i (g_{ij} + B_{ij}) D_- X^j =: \int d^2\xi d^2\theta D_+ X^i E_{ij} D_- X^j , \quad (2.8)$$

where  $X^i$  collectively denote  $(\ell, \bar{\ell}, r, \bar{r})$ ,  $g$  is a metric and  $B$  an antisymmetric tensorfield.

The (1,1) algebra is simply

$$D_+^2 = \iota \partial_{++} , \quad D_-^2 = \iota \partial_{--} \quad (2.9)$$

## 2.3 Generalized Kähler geometry

The bi-hermitean geometry of [5] was reformulated as GKG in [2]. Briefly, it is characterized by a manifold  $M$  carrying two complex structures  $J_{(+)}$  and  $J_{(-)}$

$$J_{(\pm)}^2 = -\mathbb{1} , \quad (2.10)$$

and a metric  $g$  which is hermitean with respect to both of these

$$J_{(\pm)}^t g J_{(\pm)} = g . \quad (2.11)$$

Further, the complex structures are covariantly constant with respect to  $\Gamma^{(+)}$  and  $\Gamma^{(-)}$  respective, two metric connections with torsion:

$$\begin{aligned} \nabla^{(\pm)} J_{(\pm)} &= 0 \\ \Gamma^{(\pm)} &= \Gamma^0 \pm \frac{1}{2} g^{-1} H , \quad H = dB . \end{aligned} \quad (2.12)$$

Here  $\Gamma^0$  is the usual Levi-Civita connection and  $B$  is identified with the  $B$ -field accompanying the metric in  $E$ . The geometric data can thus be taken to be the set  $(M, J_{(\pm)}, g, H)$ , but there are several other ways of characterizing GKG; see [6] for a discussion of this in relation to sigma models in different superspaces.

There are certain features of the semichiral (symplectic) description of GKG that we shall need. Defining the matrix of derivatives of  $K$  to be  $K_{LR}$ :

$$K_{LR} := \begin{pmatrix} K_{\ell r} & K_{\ell \bar{r}} \\ K_{\bar{\ell} r} & K_{\bar{\ell} \bar{r}} \end{pmatrix} \quad (2.13)$$

we find the symplectic form

$$\Omega := \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix}. \quad (2.14)$$

Using this and the complex structures, we recover the metric and  $B$ -field according to

$$g = \Omega[J_{(+)}, J_{(-)}], \quad B = \Omega\{J_{(+)}, J_{(-)}\}. \quad (2.15)$$

Since  $\Omega$  is symplectic, we see that  $H = dB = 0$  when

$$\{J_{(+)}, J_{(-)}\} = 2c \mathbb{1}, \quad (2.16)$$

with  $c$  constant. (The factor two is inserted for convenience.) Under this condition the torsion vanishes and the geometry is (pseudo)hyperkähler [7]. This can be explicitly verified from the following set of  $SU(2)$  worth of (pseudo-) complex structures  $J^{(1)}, J^{(2)}, J^{(3)}$ ,

$$J^{(1)} := \frac{1}{\sqrt{1-c^2}} (J_{(-)} + cJ_{(+)}) ,$$

$$J^{(2)} := \frac{1}{2\sqrt{1-c^2}} [J_{(+)}, J_{(-)}] , \quad (2.17)$$

$$J^{(3)} := J_{(+)} . \quad (2.18)$$

For  $|c| < 1$  the geometry is hyperkähler, while for  $|c| > 1$  the geometry is pseudo-hyperkähler [8].

### 3. Additional supersymmetry

The question of under which conditions a semichiral sigma model can have extra supersymmetries has been discussed for general targetspace dimensions in [9], and for  $4d$  target space in [10]. In the latter paper the following ansatz for the extra supersymmetry is made:

$$\begin{aligned} \delta \ell &= \bar{\varepsilon}^+ \bar{\mathbb{D}}_+ f(L, R) + g(\ell) \bar{\varepsilon}^- \bar{\mathbb{D}}_- \ell + h(\ell) \varepsilon^- \mathbb{D}_- \ell , \\ \delta \bar{\ell} &= \varepsilon^+ \mathbb{D}_+ \bar{f}(L, R) + \bar{g}(\bar{\ell}) \varepsilon^- \mathbb{D}_- \bar{\ell} + \bar{h}(\bar{\ell}) \bar{\varepsilon}^- \bar{\mathbb{D}}_- \bar{\ell} , \\ \delta r &= \bar{\varepsilon}^- \bar{\mathbb{D}}_- \tilde{f}(L, R) + \tilde{g}(r) \bar{\varepsilon}^+ \bar{\mathbb{D}}_+ r + \tilde{h}(r) \varepsilon^+ \mathbb{D}_+ r , \\ \delta \bar{r} &= \varepsilon^- \mathbb{D}_- \tilde{\bar{f}}(L, R) + \tilde{\bar{g}}(\bar{r}) \varepsilon^+ \mathbb{D}_+ \bar{r} + \tilde{\bar{h}}(\bar{r}) \bar{\varepsilon}^+ \bar{\mathbb{D}}_+ \bar{r} . \end{aligned} \quad (3.1)$$

This ansatz does not include central charge transformations which have been seen to be important in similar context, e.g., in [11]. However, the transformations in (3.1) can only give an

on-shell algebra, so central charges will be irrelevant. To see that one is forced on-shell, note that supersymmetry requires

$$[\delta_1^{(+)}, \delta_2^{(+)}]\ell = i\varepsilon_2^+ \bar{\varepsilon}_1^+ \partial \ell, \quad (3.2)$$

whereas (3.1) gives

$$\begin{aligned} [\delta_1, \delta_2]\ell = & \\ & -\varepsilon_2^+ \bar{\varepsilon}_1^+ (|f_{\bar{\ell}}|^2 i\partial_{++}\ell + (f_{\bar{\ell}}\bar{f}_r + f_r\bar{h})\bar{\mathbb{D}} + \mathbb{D}_+ r + \dots) \\ & + \bar{\varepsilon}_2^- \varepsilon_1^- (-gh)i\partial_{-}\ell + \dots \end{aligned} \quad (3.3)$$

Since  $|f_{\bar{\ell}}|^2 > 0$  we will need relations between the fields, i.e. the field equations.

A further result of [10] is that  $gh = -1$ , which is shown to imply that (2.16) is satisfied with  $c$  a constant, i.e., that the geometry is hyperkähler. Although not conclusive, the evidence strongly suggests that a manifest additional supersymmetry of a semichiral model requires a hyperkähler target space. The present work is motivated by the existence of a model that seemingly constitutes a counterexample to this conclusion; the  $SU(2) \otimes U(1)$  WZW model in semichiral coordinates.

## 4. Duality

### 4.1 The $SU(2) \otimes U(1)$ WZW model in semichiral coordinates

There is a generalized Kähler potential describing the  $SU(2) \otimes U(1)$  WZW model that involves chiral and twisted chiral superfields. It reads [12]

$$K = -\ln \hat{\chi} \ln \hat{\bar{\chi}} = \int^{\frac{\hat{\phi}\hat{\bar{\phi}}}{\hat{\chi}\hat{\bar{\chi}}}} dq \frac{\ln(1+q)}{q}, \quad (4.1)$$

and satisfies Laplace's equation

$$K_{\hat{\phi}\hat{\bar{\phi}}} + K_{\hat{\chi}\hat{\bar{\chi}}} = 0, \quad (4.2)$$

which is the necessary condition for (4,4) supersymmetry [5]. In addition there is a simple transformation to new chiral and twisted chiral coordinates,  $\phi = \ln \hat{\phi}$  and  $\chi = \ln \hat{\chi}$ , where the model is easily dualized:

$$K \rightarrow K = \frac{1}{2}(\chi - \bar{\chi})^2 + \alpha(\chi - \bar{\chi})(\phi - \bar{\phi}) + \int^{\phi + \bar{\phi} - \chi - \bar{\chi}} dq \ln(1 + e^q). \quad (4.3)$$

In the new coordinates, the condition for (4,4) supersymmetry becomes

$$e^{-(\phi + \bar{\phi})} K_{\phi\bar{\phi}} + e^{-(\chi + \bar{\chi})} K_{\chi\bar{\chi}} = 0. \quad (4.4)$$

To facilitate dualization, the  $K$  in (4.3) is the transformed potential modulo generalized Kähler gauge transformations<sup>1</sup>, and added the  $\alpha$  term which represents a constant  $B$  field, and hence will not change the geometry (which is torsionful). The corresponding action has the shift symmetry

$$\phi \rightarrow \phi + \lambda, \quad \chi \rightarrow \chi + \lambda, \quad (4.5)$$

<sup>1</sup>Transformations of  $K$  that do not affect the field equations.

which we gauge using the Large Vector Multiplet (LVM) [13]. This means that we couple the model to the three vector fields  $V^\phi, V^\chi$  and  $V'$  and then introduce Lagrange multipliers  $X_\phi, X_\chi$  and  $X'$  that are linear combinations of semichiral fields and set the fieldstrengths of the LVM to zero. They read

$$\begin{aligned} X_\phi &= \frac{i}{2}(\ell - \bar{\ell} - r + \bar{r}) \\ X_\chi &= \frac{i}{2}(-\ell + \bar{\ell} - r + \bar{r}) \\ X' &= \frac{1}{2}(\ell + \bar{\ell} - r - \bar{r}) . \end{aligned} \quad (4.6)$$

It is convenient to redefine the LVM fields by a gauge transformation. Dualization is then the Legendre transformation  $(V^\phi, V^\chi, V') \rightarrow (X_\phi, X_\chi, X')$  obtained from

$$-\frac{1}{2}(V^\chi)^2 - \alpha V^\chi V^\phi + \int^{V'} dq \ln(1 + e^q) - V' X' - V^\phi X_\phi - V^\chi X_\chi , \quad (4.7)$$

and results in the semichiral action (derived slightly differently in [14], and discussed in [15]):

$$-\frac{1}{2\alpha^2} X_\phi^2 + \frac{1}{\alpha} X_\phi X_\chi - \int^{X'} dq \ln(e^q - 1) . \quad (4.8)$$

We expect the potential in (4.8) to correspond to a semichiral model with (4,4) supersymmetry.

## 4.2 Hyperkähler duals

Now, the semichiral dual of a (twisted) chiral model with torsion and (4,4) supersymmetry can be a (torsionless) hyperkähler model when the target space is  $4d$  if the original model has the LVM isometry in the coordinates where the Laplace equation takes the form (4.2). Namely, consider

$$\begin{aligned} K &= K(-\iota(\phi - \bar{\phi}), -\iota(\chi - \bar{\chi}), \phi + \bar{\phi} - \chi - \bar{\chi}) =: K(x, y, z) \\ K_{\phi\bar{\phi}} + K_{\chi\bar{\chi}} &= 0 \iff K_{xx} + K_{yy} + 2K_{zz} = 0 . \end{aligned} \quad (4.9)$$

As before, we may dualize to semichirals  $(x, y, z) \rightarrow (u, v, w)$  using

$$K(V^x, V^y, V^z) - uV^x - vV^y - wV^z . \quad (4.10)$$

For ease of notation in what follows we use  $(V^x, V^y, V^z)$  in place of  $(V^\phi, V^\chi, V')$  and  $(u, v, w)$  for  $(X_\phi, X_\chi, X')$ . The dual (Legendre transformed) semichiral potential  $\tilde{K}$  is found by integrating out the LVM  $V$ 's and solving them in terms of  $(u, v, w)$ :

$$\begin{aligned} \tilde{K}(u, v, w) &= K(V^x(u, v, w), V^y(u, v, w), V^z(u, v, w)) \\ &\quad - uV^x(u, v, w) - vV^y(u, v, w) - wV^z(u, v, w) . \end{aligned} \quad (4.11)$$

Using the standard inverse relation that results from a Legendre transform we have

$$K_{ab} = -(\tilde{K}_{\hat{a}\hat{b}})^{-1} =: -\tilde{K}^{\hat{a}\hat{b}} , \quad (4.12)$$

where the unhatted indices represent the original coordinates and the hatted represent the dual. Thus the condition for (4,4) supersymmetry, the Laplace equation, in (4.9) becomes

$$K_{xx} + K_{yy} + 2K_{zz} = \tilde{K}^{uu} + \tilde{K}^{vv} + 2\tilde{K}^{ww} = 0 . \quad (4.13)$$

This is in fact the form of hyperkähler condition (2.16) with  $c = 0$  takes in the  $(u, v, w)$  coordinates. In the original left and right coordinates, the only nontrivial condition that results from (2.16) in  $4d$  is

$$(1+c)|\tilde{K}_{\ell r}|^2 + (1-c)|\tilde{K}_{\ell \bar{r}}|^2 = 2\tilde{K}_{\ell \bar{\ell}}\tilde{K}_{r \bar{r}}. \quad (4.14)$$

In  $(u, v, w)$  coordinates this reads

$$\begin{aligned} & 2\tilde{K}_{uv}^2 + \tilde{K}_{uw}^2 + \tilde{K}_{vw}^2 + c(\tilde{K}_{uw}^2 - \tilde{K}_{vw}^2 - \tilde{K}_{uu}\tilde{K}_{ww} + \tilde{K}_{vv}\tilde{K}_{ww}) \\ & = 2\tilde{K}_{uu}\tilde{K}_{vv} + \tilde{K}_{uu}\tilde{K}_{ww} + \tilde{K}_{vv}\tilde{K}_{ww}. \end{aligned} \quad (4.15)$$

It is a remarkable fact that *this is equivalent to*<sup>2</sup>

$$\tilde{K}^{uu} + \tilde{K}^{vv} + 2\tilde{K}^{ww} = 4c(\tilde{K}^{uu} - \tilde{K}^{vv}), \quad (4.16)$$

so (4.13) is satisfied (for  $c = 0$ ). The relation (4.16) may be verified using the following formula for the inverse of a  $3 \times 3$  matrix:

$$\begin{aligned} & \begin{pmatrix} \tilde{K}_{uu} & \tilde{K}_{uv} & \tilde{K}_{uw} \\ \tilde{K}_{vu} & \tilde{K}_{vv} & \tilde{K}_{vw} \\ \tilde{K}_{wu} & \tilde{K}_{wv} & \tilde{K}_{ww} \end{pmatrix}^{-1} \\ & = \frac{1}{\Delta} \begin{pmatrix} \tilde{K}_{vv}\tilde{K}_{ww} - \tilde{K}_{vw}^2 & \tilde{K}_{vw}\tilde{K}_{uw} - \tilde{K}_{vu}\tilde{K}_{ww} & \tilde{K}_{vu}\tilde{K}_{vw} - \tilde{K}_{vv}\tilde{K}_{uw} \\ \tilde{K}_{vw}\tilde{K}_{uw} - \tilde{K}_{uv}\tilde{K}_{ww} & \tilde{K}_{uu}\tilde{K}_{ww} - \tilde{K}_{uw}^2 & \tilde{K}_{uv}\tilde{K}_{uw} - \tilde{K}_{uu}\tilde{K}_{vv} \\ \tilde{K}_{vu}\tilde{K}_{wv} - \tilde{K}_{vv}\tilde{K}_{uw} & \tilde{K}_{uv}\tilde{K}_{uw} - \tilde{K}_{uu}\tilde{K}_{vw} & \tilde{K}_{uu}\tilde{K}_{vv} - \tilde{K}_{uv}^2 \end{pmatrix}, \end{aligned} \quad (4.17)$$

where  $\Delta$  denotes the determinant of the original matrix.

The conclusion of this subsection is thus that if a (twisted) chiral  $4d$  sigma model satisfied the Laplace equation *and* has LVM (shift) isometry in these coordinates, it has semichiral dual which is hyperkähler.

This explains the geometry underlying the hyperkähler constructions of [16].

## 5. Extended supersymmetry in $(1, 1)$ superspace

The conclusions in subsec.4.2 do not apply to our model in (4.8). In the coordinates where the Laplace equation takes the form (4.2), the LVM isometry is not realized as a translation symmetry, and conversely, in the coordinates where the LVM isometry is realized as translations the condition for extended supersymmetry takes the form (4.4). Indeed this model has torsion, as can be directly checked and it has extended supersymmetry inherited from the potential (4.3).

To address the question of how the additional supersymmetry is realized it is useful to descend to  $(1, 1)$  superspace, in light of the negative results in section.3.

<sup>2</sup>In coordinates  $s := (u+v)/2$  and  $t := -(u-v)/2$  the condition becomes even simpler:  $\tilde{K}^{ss} + \tilde{K}^{tt} + \tilde{K}^{ww} \propto c\tilde{K}^{st}$ .

### 5.1 (1, 1) keeping the auxiliary spinors

The potential (2.6) results in the following (1, 1) sigma model Lagrangian

$$\mathcal{L} = D_+ X^i E_{ij}(X) D_- X^j + \Psi_+^R K_{RL} \Psi_-^L := \mathcal{L}_1 + \mathcal{L}_2, \quad (5.1)$$

where the  $\Psi$ s are related to the components in (2.7) as

$$\begin{aligned} \Psi_+^r &:= \psi_+ - D_+ X^A J_{(+A}^r \\ \Psi_-^\ell &:= \psi_- - J_{(-A}^\ell D_- X^A, \end{aligned} \quad (5.2)$$

and their complex conjugate ( $R =: (r, \bar{r})$ ,  $L =: (\ell, \bar{\ell}) \Rightarrow i = (L, R)$ ). Integrating out the auxiliary spinors sets  $\Psi = 0$ , but here we keep them and investigate their role in extended supersymmetry transformations forming an  $SU(2)$  algebra<sup>3</sup>. Explicitly, the extended supersymmetries are generated by complex structures [5]

$$I_{(+)}^{(\mathfrak{a})} I_{(+)}^{(\mathfrak{b})} = -\delta^{\mathfrak{a}\mathfrak{b}} + \varepsilon^{\mathfrak{a}\mathfrak{b}\mathfrak{c}} I_{(+)}^{(\mathfrak{c})} \quad (5.3)$$

with  $\mathfrak{a} = 1, 2, 3$  and the identification  $I_{(+)}^{(3)} := J_{(+)}$ . Starting from one of the complex structures  $I^{(\mathfrak{a})} =: I_j^i$  and assuming that the corresponding transformation of  $X^i$  yields a symmetry of  $\mathcal{L}_1$ , we ask under what condition the transformation can be extended to include  $\Psi$  compatible with (5.2) and leaving the full Lagrangian invariant (up to total derivatives). At the (2, 2) level, such a transformation would correspond to a transformation of the semichiral fields either leading to hyperkähler geometry or somehow missed in our (2, 2) analysis.

A general ansatz for the non manifest (+)-supersymmetry is (dropping the (+) index on  $I$ ):

$$\delta X^i = \varepsilon^+ [I_j^i D_+ X^j + M^i_R \Psi_+^R],$$

where  $M$  is a  $4 \times 4$  matrix with  $M_L^i = 0$ . The  $\Psi$ -transformations follows from transforming both sides in (5.2) and read

$$\begin{aligned} \delta \Psi_+^r &= \varepsilon^+ \left[ (I_r^i - [M, J_{(+)}]_r^i) D_+ \Psi_+^r + (I_{j,r}^i + \mathcal{M}(M, J_{(+)})_{rj}^i) D_+ X^j \Psi_+^r \right. \\ &\quad \left. - M_{r,r'}^i \Psi_+^{r'} \right] \\ \delta \Psi_-^\ell &= \varepsilon^+ \left[ -[M, J_{(-)}]_r^\ell D_- \Psi_-^\ell + \mathcal{M}(M, J_{(-)})_{rj}^\ell D_- X^j \Psi_-^\ell \right. \\ &\quad + (I_\ell^i - M_{r,j}^\ell J_{(+)\ell}^i) D_+ \Psi_-^\ell + (I_{j,\ell}^i - M_{r,j}^\ell J_{(+)\ell}^i) D_+ X^j \Psi_-^\ell \\ &\quad \left. - M_{r,\ell}^i \Psi_-^r + \left( [I, J_{(-)}]_j^\ell - M_r^\ell [J_{(+), J_{(-)}]_j^r \right) \nabla_+^{(-)} D_- X^j \right], \end{aligned} \quad (5.4)$$

and their complex conjugate expressions. Here overdot denotes the free index, covariant derivatives are defined in (2.12) and the Magri-Morosi concomitant for two endomorphisms  $I$  and  $J$  reads [17]

$$\mathcal{M}(I, J)_{BD}^A = I_B^F J_{D,F}^A - J_D^F I_{B,F}^A - I_F^A J_{D,B}^F + J_F^A I_{B,D}^F. \quad (5.5)$$

<sup>3</sup>Corresponding to (+)-supersymmetries. The general case also involves (-)-supersymmetries.



The algebra of transformations (5.4) and (5.4) close on-shell where  $\Psi = 0$ ,  $\nabla_+^{(-)} D_- X^j = 0$  and the  $X$  transformations close. The issue is invariance of the action.

Invariance of the action requires  $M$  to satisfy a number of conditions which we now list: First, raising and lowering indices on  $M$  with  $K_{RL}$ <sup>4</sup>,

$$\begin{aligned}
M_{L[R,\dot{R}]} - M_{[R\dot{R}],L} &= 0 \\
M_{[R\dot{R}]} &= -\frac{1}{2} K_{\dot{R}L} [I_{(+), J_{(-)}]_j^L g^{jL} K_{LR} \\
M_{\dot{R}}^R &= -\frac{1}{2} K_{\dot{R}L} [I_{(+), J_{(-)}]_j^L g^{jR} \\
-K_{(\dot{R}|L} [M, J_{(-)}]_{R)}^L &= [J, M]_{(\dot{R}R)} + C_{(\dot{R}|R} M_{R)}^R = 0 \\
K_{[\dot{R}|L} \mathcal{M}(M, J_{(-)})_{R]j}^L D_- X^j &= -\frac{1}{2} D_- (K_{[\dot{R}|L} [M, J_{(-)}]_{R]}^L) \\
K_{LR} I_{\dot{R}}^R - K_{\dot{R}L} I_L^L + K_{LR} [\tilde{J}, M]_{\dot{R}}^R + C_{LL} K^{LR} M_{[R\dot{R}]} &= 0, \\
(I_{j,\dot{R}}^R + \mathcal{M}(M, J_{(+)} )_{\dot{R}j}^R) K_{RL} + K_{\dot{R}L} I_j^k + K_{\dot{R}L} (I_{j,L}^L - M_{R(+),j,L}^R J_{R(+),L}^R) \\
&= ((I_L^L - M_{R(+),L}^R J_{R(+),L}^R) K_{\dot{R}L})_{,j}, \tag{5.6}
\end{aligned}$$

where  $(\tilde{J})_{\dot{R}}^R := K^{RR'} J K_{R'\dot{R}}$  and we have used (5.6) and the explicit form of  $J_{(+)}$  from [7] in the last equation, as well as the definitions

$$C_{LL} := [J, K]_{LL} = 2i \begin{pmatrix} 0 & K_{\ell,\bar{\ell}} \\ -K_{\bar{\ell}\ell} & 0 \end{pmatrix} = -2K_{\ell,\bar{\ell}} \sigma_2, \quad C_{RR} = -2K_{r,\bar{r}} \sigma_2, \tag{5.7}$$

with the relations involving the Pauli matrix being particular to  $4d$  target space.

The set of conditions (5.6) looks forbidding. Nevertheless we know that it has to be satisfied by  $I = J_{(+)}$ , corresponding to the second supersymmetry which is manifest in  $(2,2)$  superspace. Indeed it is, as seen below.

The conditions are also expected to be satisfied for some hyperkähler manifolds, as discussed in sec.3. Testing this on the hyperkähler complex structures in (2.17), we find that the relations (5.6) determine  $M$  in the three cases according to

$$\begin{aligned}
J^{(3)} : M_{\dot{R}}^R &= \delta_{\dot{R}}^R, \quad M_{[R\dot{R}]} = 0 \\
J^{(1)} : M_{\dot{R}}^R &= \frac{c \delta_{\dot{R}}^R}{\sqrt{1-c^2}}, \\
J^{(2)} : M_{\dot{R}}^R K_{LR} &= -\frac{1}{\sqrt{1-c^2}} K_{\dot{R}L} J_{(-)L}^L = -\frac{1}{\sqrt{1-c^2}} J K_{\dot{R}L} \\
M_{[R\dot{R}]} &= -\frac{1}{\sqrt{1-c^2}} K_{\dot{R}L} J_{(-)R}^L = -\frac{1}{\sqrt{1-c^2}} C_{\dot{R}R} \tag{5.8}
\end{aligned}$$

Each case satisfies the relation in (5.6) (provided that  $c$  is constant), again using the relations for  $J_{(\pm)}$  from [7]. We now want to test the relations on our semichiral version of  $SU(2) \otimes U(1)$ .

<sup>4</sup>This is just a notational convenience and does not necessarily identify  $K_{RL}$  as a metric

## 5.2 The additional complex structures for $SU(2) \otimes U(1)$

The additional supersymmetries of (4.1) may be found in [5]. In the new coordinates where the potential is (4.3) they read

$$\begin{aligned}\delta\phi &= e^{\bar{\chi}-\phi}\bar{\epsilon}^+\bar{\mathbb{D}}_+\bar{\chi} + e^{\chi-\phi}\bar{\epsilon}^-\bar{\mathbb{D}}_-\chi \\ \delta\bar{\phi} &= e^{\chi-\bar{\phi}}\epsilon^+\mathbb{D}_+\chi + e^{\bar{\chi}-\bar{\phi}}\epsilon^-\mathbb{D}_-\bar{\chi} \\ \delta\chi &= -e^{\bar{\phi}-\chi}\bar{\epsilon}^+\mathbb{D}_+\bar{\phi} - e^{\phi-\chi}\epsilon^-\mathbb{D}_-\phi \\ \delta\bar{\chi} &= -e^{\phi-\bar{\chi}}\epsilon^+\mathbb{D}_+\phi - e^{\bar{\phi}-\bar{\chi}}\bar{\epsilon}^-\mathbb{D}_-\bar{\phi} .\end{aligned}\tag{5.9}$$

These relations survive in the  $(1,1)$  reduction with  $\mathbb{D}_\pm \rightarrow D_\pm$ . From the general formula of the supersymmetry transformation of a field  $\varphi$  and (5.9) we read off the additional complex structures:

$$\delta\varphi = \frac{1}{2} \left[ \left( I_{(\pm)}^{(1)} + iI_{(\pm)}^{(2)} \right) \epsilon^\pm D_\pm \varphi + \left( I_{(\pm)}^{(1)} - iI_{(\pm)}^{(2)} \right) \bar{\epsilon}^\pm D_\pm \varphi \right] .\tag{5.10}$$

For the  $I_{(+)}^{(\mathfrak{a})}$  we find (in a  $(\phi, \chi, \bar{\phi}, \bar{\chi})$  basis )

$$I_{(+)}^{(\mathfrak{a})} = \begin{pmatrix} 0 & \mathbb{A}^{(\mathfrak{a})} \\ -(\mathbb{A}^{(\mathfrak{a})})^{-1} & 0 \end{pmatrix}\tag{5.11}$$

for  $\mathfrak{a} = 1, 2$ , with

$$\begin{aligned}\mathbb{A}^{(1)} &= \begin{pmatrix} 0 & e^{\bar{\chi}-\phi} \\ e^{\chi-\bar{\phi}} & 0 \end{pmatrix} \\ \mathbb{A}^{(2)} &= \begin{pmatrix} 0 & ie^{\bar{\chi}-\phi} \\ -ie^{\chi-\bar{\phi}} & 0 \end{pmatrix} ,\end{aligned}\tag{5.12}$$

and with  $I_{(+)}^{(3)} = J$ , the canonical complex structure. To find these complex structures on the semichiral side, we note that unlike the general case for duality, at the  $(1,1)$  level, there must be a coordinate transformation that take us from the (twisted) chiral coordinates to the semichiral coordinates, since both coordinatize the same geometric object.

A  $(1,1)$  superspace analysis of the duality (4.3) to (4.8) yields the transformation [1]

$$\begin{aligned}i(\bar{\phi} - \phi) &= -i(r - \bar{r}) \\ i(\bar{\chi} - \chi) &= \frac{1}{2}(\ell - \bar{\ell} - r + \bar{r}) \\ \phi + \bar{\phi} - \chi - \bar{\chi} &= \ln(e^{\frac{1}{2}(\ell + \bar{\ell} - r - \bar{r})} - 1) .\end{aligned}\tag{5.13}$$

This transformation is not complete, however. One more relation is needed. As described in [1], it turns out to be most convenient to identify the full coordinate transformation not in  $(L, R)$  semichiral coordinates, but in so called  $(X, Y)$  coordinates where the  $J_{(+)}$  derived from the semichiral model is canonical [7] [16]. We require that it stays invariant under the coordinate transformation and thus is mapped to the  $J_{(+)}$  in the (twisted) chiral model. As discussed in [1], this condition is

natural but not necessarily the only possibility. In [18] the transformations of complex structures under T-duality is described. The transformed complex structures are obtained by finding the supersymmetry for the vector fields in the first order action for the  $(1,1)$  dualization. Using this method directly is difficult, since the relation to the  $(2,2)$  coordinate fields is needed as additional data. It should also be noted that the derivation is not directly applicable in the present case where the first order action contains more vector fields and auxiliaries. A modified derivation can presumably be worked out using the T-duality relations in [13]. Here it is sufficient to note that the present route yields the extended supersymmetry of the semichiral model corresponding to that of the (twisted) chiral model.

For  $a = 1, 2$ , the expression for  $I_{(+)}^{(a)}$ , in  $(L, R)$  coordinates turns out to be<sup>5</sup>

$$I_{(+)}^{(a)} = \frac{1}{4N} \left\{ \begin{pmatrix} E & -M \\ e^{-X} & -e^{-X}E \end{pmatrix} \otimes \mathbb{A}^{(a)} + \begin{pmatrix} M & -ME \\ e^{-X}E & -e^{-X}E^2 \end{pmatrix} \otimes \mathbb{A}^{(a)} \sigma_1 \right. \\ \left. + \frac{N}{e^X} \left[ \begin{pmatrix} 0 & 0 \\ 1 & -E \end{pmatrix} \otimes \bar{\mathbb{A}}^{(a)} + \begin{pmatrix} 0 & 0 \\ E & -1 \end{pmatrix} \otimes \bar{\mathbb{A}}^{(a)} \sigma_1 \right] \right\}, \quad (5.14)$$

where

$$-\frac{1}{4N} := K_{\ell\bar{\ell}}, \quad -\frac{E}{4N} := K_{\ell\ell}, \quad \frac{M}{4N} := K_{r\bar{r}}, \quad (5.15)$$

$K$  is given in (4.8) and  $X$  is defined in (4.6) (dropping the prime).

As explained in [13], these  $I_{(+)}^{(a)}$  give matrices  $M^i_j$  that do not satisfy the conditions (5.6) and hence the corresponding transformations cannot be lifted to transformations on semichiral fields.

## 6. Conclusions

Using a novel  $(1,1)$  formulation of semichiral sigma models where the auxiliary spinors are kept, conditions for extra supersymmetries that can be lifted to a  $(2,2)$  semichiral formulation are derived. For a  $4d$  target space these conditions are shown to be met in supersymmetries generated by a hyperkähler set of complex structures including the second supersymmetry generated by  $J_{(+)}$ . This agrees with the  $(2,2)$  result that hyperkähler is an allowed targetspace geometry when extra supersymmetries are realized as transformations of the semichiral fields [9]. In the paper [9] it is also argued that the  $SU(2) \otimes U(1)$  WZW model shows that hyperkähler is not exhaustive, since there is a (twisted) chiral formulation with manifest  $(4,4)$  which has a semichiral dual.

To investigate the counterexample, we derive the extra right complex structures and map them to the semichiral coordinates in the  $(1,1)$  formulation, thus displaying the extra right supersymmetries for the semichiral model for the first time (for the  $(1,1)$  sigma model with auxiliary spinors integrated out). We then derive the transformations of the auxiliary spinors for this case and check if the symmetry can be lifted to transformations on semichirals in  $(2,2)$  superspace. The result is negative even for this case of extra right transformations only.

The resolution of the counterexample conundrum is thus that *the extra supersymmetries are there in the  $(1,1)$  formulation of the semichiral model with zero auxiliary spinors, but are incompatible with nonzero auxiliaries*. Another way of saying this at the  $(2,2)$  level is that the chirality

<sup>5</sup>The full coordinate transformation is given in [13].

conditions on the semichiral fields is incompatible with a realization of the extra supersymmetries as transformations of those fields. Presumably additional  $(2,2)$  auxiliary fields are needed.

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